MATHS FIELD GUIDE · GRADE 9

The sum of the first n natural numbers

OPENING PROBLEM

Find the sum of the forward row.

Forward	2	5	8	11
Reverse	11	8	5	2
Pair	13	13	13	13

02 Pair the ends to find a sum
03 Rules and worked cases
04 From the givens to the answer
05 Distinguish a term from a total
06 Check your reasoning
07 Check your reasoning

ILLUSTRATED PROBLEM SHEET

Pair the ends to find a sum

Forward and reverse copies of an AP have equal column sums. Divide their combined total by two.

1

Forward	2	5	8	11
Reverse	11	8	5	2
Pair	13	13	13	13

Find the sum of the forward row.

Four pairs give 52, twice the sum. The sum is 26.

2

$$1 + 2 + \cdots + 20$$
$$\downarrow$$
$$20 \times 21 \div 2$$
$$\downarrow$$
$$210$$

Which two terms form the first pair?

1 and 20. Every pair of opposite positions totals 21.

3

$$4 + 7 + 10 + 13 + 16$$
$$\downarrow$$
$$2 \times 20 + 10$$
$$\downarrow$$
$$50$$

What if one middle term is left over?

Add the two outer pairs and then the middle 10. The average is 10.

4

$$S_{10} = 190$$
$$\downarrow$$
$$S_9 = 153$$
$$\downarrow$$
$$a_{10} = 37$$

How can two totals reveal just one term?

The first nine terms occur in both totals. Subtraction leaves the tenth alone.

Five equally spaced terms
total = five times the middle term.

Six equally spaced terms
total = three times the sum of the endpoints.

CHAPTER PROBLEM

Find 1 + 2 + ... + 10.

55. 10 × 11 / 2 = 55.

MATHS REFERENCE · GRADE 9

The sum of the first n natural numbers

Rules, conditions and worked cases

Recognise the case	Mathematical reasoning
Pair opposite ends	1 + 2 + … + 10 has five pairs of 11, giving 55.
Odd number of terms	1 through 9 has four pairs of 10 and a middle 5, totalling 45.
Subtract two totals	11 + … + 20 = 20 × 21/2 − 10 × 11/2 = 155.

Read the representations

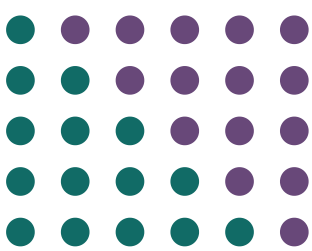
Successive rows form a triangular count

Build rows containing 1, 2, 3, …, n objects. Their total is S = 1 + 2 + … + n.

A reversed copy of the arrangement fits against the first copy.

Together the two copies make n rows of n + 1 objects, so 2S = n(n + 1).

Two colours: two copies of the triangular count



5 rows × 6 dots = twice (1 + 2 + 3 + 4 + 5)

With n = 5, two triangular arrangements make a 5 by 6 rectangle. The general argument works for every positive integer n.

Why divide by two?

The rectangle counts the original arrangement twice.

Therefore S = n(n + 1)/2.

One of consecutive integers n and n + 1 is even, so the result is always an integer.

3 7 11 15 ?

What is the next term if the rule is add four?

Nineteen. Each gap is four.

WORKED EXAMPLES

From the givens to the answer

Does n(n + 1) already give the sum?

A learner forgets the final division by two.

1. That product counts two copies of the rows.
2. One original triangular arrangement is half of the combined rectangle.
3. Divide by 2 exactly once.

Find 1 + 2 + ... + 20

1. Here n = 20 because the final natural number is 20.
2. S = 20 × 21 / 2 = 10 × 21 = 210.
3. Check the size: 20 terms with average 10.5 give 210.

Count handshakes among 8 people, each pair once

1. Person 1 can greet 7 others. The next person has 6 new partners, then 5, and so on.
2. The total is 7 + 6 + ... + 1 = 7 × 8 / 2 = 28.
3. Using 8 × 7 without dividing would count each pair twice.

Check another case

Example 1

Find 6 + 7 + ... + 15.

105

Sum 1 through 15 is 120; subtract sum 1 through 5, which is 15.

Example 2

Find the sum of the first 16 natural numbers.

136

16 × 17 / 2 = 136.

Position	1	2	3	4
Term	3	7	11	15

How do you reach term 4 straight from term 1?

Start at 3 and add three jumps of 4. The result is 15.

ILLUSTRATED PROBLEM SHEET

Distinguish a term from a total

In an arithmetic progression, $a_n = a_1 + (n - 1)d$.

1

7 11 15 19 23

What do the four gaps show?

The common difference is 4. Reaching term 5 takes four jumps.

2

$a_1 = 7$ and $d = 4$

↓

$a_{12} = 7 + 11 \times 4$

↓

51

Why use eleven instead of twelve?

The first term is already present before making any jumps.

3

$7 + 4(n - 1) = 39$

↓

$4(n - 1) = 32$

↓

$n = 9$

Does 39 occur?

Yes. The position is a positive integer. The adjacent terms are 35 and 43.

4

Position	3	8
Term	15	35

Recover the difference without knowing term 1.

Five gaps add 20, so d = 4. Step back twice from 15 to get 7.

$d = 0$
a constant AP.

$d < 0$
each term is smaller than its predecessor.

CHAPTER PROBLEM

Find the sum of the first 12 natural numbers.

78. Pair forward and reverse counts. Half of 12×13 gives 78.

QUESTION BANK · FOLD OR COVER THE RIGHT COLUMN

Check your reasoning

Try the problem	Solution and reasoning
01 The first n natural numbers sum to 55. Find n.	10 × 11 ÷ 2 = 55. The sums strictly increase for positive n, so n = 10.
02 Why is n(n + 1)/2 an integer? A) Both factors are always even B) One of two consecutive integers is even C) n is always even D) Division by 2 never changes a number	One of two consecutive integers is even Every consecutive pair contains an even integer.
03 Is 40 a term of this progression?	No. Its difference from 7 is 33, which is not a whole number of four-unit gaps.
04 A triangular arrangement has 9 rows containing 1 through 9 dots. Which expression counts it? A) 9^2 B) 9 × 10 C) 9 × 10 / 2 D) 10 × 11 / 2	9 × 10 / 2 The paired rectangle has 9 rows of 10, then one copy is half.
05 How many distinct handshakes occur among 6 people if each pair shakes hands once?	15 5 + 4 + 3 + 2 + 1 = 15.
06 Which claim correctly describes two copies of 1 + … + n? A) They total n^2 B) They total n(n + 1) C) They total n + 1 D) They total n(n − 1)	They total n(n + 1) The paired rows form n rows of n + 1 objects.

22 17 12 7 ?

Can the jump be negative?

Yes. Subtract five each time. The next term is 2.

QUESTION BANK · FOLD OR COVER THE RIGHT COLUMN

Check your reasoning

Try the problem	Solution and reasoning
07 Explain why summing 1 through 100 does not mean multiplying 100 by 100.	The 100 terms are not all equal to 100. Pairing the first and last values gives 50 pairs of 101. The sum is 5050, half of 100 × 101.
08 Explain how to find 11 + 12 + … + 25 using two natural-number sums.	Compute the sum from 1 through 25. Subtract the sum from 1 through 10. This leaves exactly the required terms, giving 325 − 55 = 270.
09 For the sum 1 + 2 + … + 18, what is n in n(n + 1)/2? A) 1 B) 17 C) 18 D) 19	18 There are 18 natural-number terms.
10 Explain the paired-arrangement proof of the sum formula.	Start with rows of lengths 1 through n. A reversed copy creates n rows each of length n + 1. The rectangle counts two copies, so divide its n(n + 1) objects by 2.
11 Find 1 + 2 + … + 30.	465 30 × 31 / 2 = 465.

Input	1	2	3
Output	3	5	7

Which rule doubles then adds one?

Output = 2 × input + 1. Input 6 gives 13.

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