MATHS FIELD GUIDE · GRADE 9

Proving square roots irrational

OPENING PROBLEM

Why must p be even?

$$\sqrt{2} = p/q$$
$$\downarrow$$
$$p^2 = 2q^2$$
$$\downarrow$$
$$p \text{ is even}$$

02 Try the fraction and follow the contradiction
03 Rules and worked cases
04 From the givens to the answer
05 Bound a root and test the claim
06 Check your reasoning
07 Check your reasoning

ILLUSTRATED PROBLEM SHEET

Try the fraction and follow the contradiction

A rational number can be written as p/q in lowest terms, with q nonzero.

1

$$\sqrt{2} = \frac{p}{q}$$
$$\downarrow$$
$$p^2 = 2q^2$$
$$\downarrow$$
$$p \text{ is even}$$

Why must p be even?

An odd integer has an odd square. An even square can only come from an even integer.

2

$$p = 2k$$
$$\downarrow$$
$$4k^2 = 2q^2$$
$$\downarrow$$
$$q^2 = 2k^2$$

What does this imply about q?

It is even too. Both p and q divisible by 2 contradict lowest terms.

3

$$r = 3 + \sqrt{2}$$
$$\downarrow$$
$$\sqrt{2} = r - 3$$

Could r be rational?

If so, subtracting the rational number 3 would make √2 rational. That contradicts the proof.

4

$$(\sqrt{2})(\sqrt{2}) = 2$$
$$\downarrow$$
$$\sqrt{2} - \sqrt{2} = 0$$

Can two irrational inputs produce a rational result?

Yes. Both calculations do. The type of each input alone does not decide the output.

$\sqrt{18} = 3\sqrt{2}$
irrational.

$\sqrt{18} \div \sqrt{2} = 3$
rational.

CHAPTER PROBLEM

From $a^2 = 3b^2$, explain why $a = 3k$ for some integer k.

3 divides a^2. Because 3 is prime, it divides a. Therefore a is an integer multiple of 3, written $a = 3k$.

MATHS REFERENCE · GRADE 9

Proving square roots irrational

Rules, conditions and worked cases

Recognise the case	Mathematical reasoning
Set up the contradiction	Write $\sqrt{3} = a/b$ with a and b coprime, $b > 0$. Then $a^2 = 3b^2$.
Force a shared prime	3 divides a^2, hence a. Put $a = 3k$ to obtain $b^2 = 3k^2$. Now both contain 3.
Test the boundary	$\sqrt{9} = 3$. The step "9 divides a^2, so 9 divides a" fails for $a = 3$ because 9 is composite.

Read the representations

What a contradiction proof does

To prove a claim, temporarily assume its opposite.

Follow valid consequences until the assumption forces an impossibility. Then reject the assumption.

For a rational root, choose a fraction a/b in lowest terms with $b > 0$. Then a and b have no common prime factor.

1. **Assumption**$\sqrt{p}$ is a fraction a/b in lowest terms.
2. **Consequence**Prime p must divide a.
3. **Further consequence**Prime p must also divide b.
4. **Contradiction**a and b share a prime factor despite being coprime.

The contradiction must conflict with the original assumption, not merely look surprising.

A prime dividing a square divides its base

In a^2, each prime factor of a occurs twice as often.

If prime p divides a^2, p must already occur in a. Thus p divides a.

For $p = 2$ this says: an even square has an even base. For $p = 3$: a square divisible by 3 has a base divisible by 3.

$$72 = 8 \times 9$$
$$\downarrow$$
$$2^3 \times 3^2$$

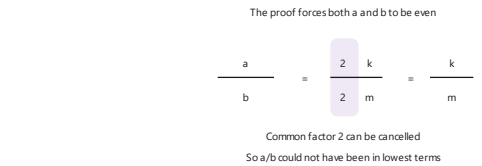
Why do two different factor trees end alike?

Their terminal prime factors, counted with multiplicity, are unique apart from order.

WORKED EXAMPLES

From the givens to the answer

Why √2 cannot equal a fraction in lowest terms



Once both are even, a = 2k and b = 2m for integers k and m. A common factor 2 can be cancelled.

1. Assume $\sqrt{2} = a/b$, with integers a, b coprime and $b > 0$.
2. Squaring gives $a^2 = 2b^2$. Thus 2 divides a^2 and therefore a.
3. Write $a = 2k$. Then $4k^2 = 2b^2$, so $b^2 = 2k^2$.
4. Thus 2 divides b as well. Both a and b share factor 2.
5. This contradicts lowest terms, so $\sqrt{2}$ is irrational.

Why √3 cannot equal a fraction in lowest terms

1. Assume $\sqrt{3} = a/b$ in lowest terms with $b > 0$.
2. Square: $a^2 = 3b^2$. Prime 3 divides a^2, so 3 divides a.
3. Write $a = 3k$. Then $9k^2 = 3b^2$, so $b^2 = 3k^2$.
4. Prime 3 also divides b. This contradicts coprime a and b.
5. Therefore $\sqrt{3}$ is irrational.

Do many decimal digits prove irrationality?

A learner computes 1.7320508 and announces a proof that $\sqrt{3}$ is irrational.

1. A finite display may be a rounded value. It cannot describe infinitely many digits.
2. A rational number can also have a long recurring expansion.
3. The contradiction proof uses exact integer divisibility and avoids this limitation.

Check another case

Example 1

Write a complete proof that $\sqrt{3}$ is irrational without looking back.

State a coprime fraction assumption and $b > 0$.

Use $a^2 = 3b^2$ to conclude $a = 3k$.

Derive $b^2 = 3k^2$ to make b divisible by 3.

Identify the contradiction and conclude.

Example 2

Explain where the $\sqrt{2}$ proof uses unique prime factorisation.

It justifies that 2 dividing a^2 implies 2 divides a.

It justifies the same inference for b.

Every prime appearing in a square comes from its integer base.

Work backwards and test the limits

Problem	Derivation
Why is "its decimal has many digits" insufficient?	The rational number 1/7 has infinitely many repeating digits. A finite display cannot show that a decimal never becomes periodic.
Prove $\sqrt{12}$ is irrational using $\sqrt{3}$.	$\sqrt{12} = 2\sqrt{3}$. If $\sqrt{12}$ were rational, dividing by 2 would make $\sqrt{3}$ rational, contradicting its prime-divisibility proof.
Could $\sqrt{8}$ equal 2.828 exactly?	No. 2.828 is a rational decimal and its square is 7.997584. It estimates the positive root of 8 without equalling it.

Number	Power of 2	Power of 3
24	3	1
36	2	2

Read the HCF from the exponents.

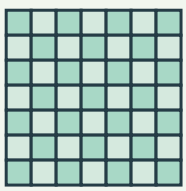
Take the smaller exponent in each column. HCF = $2^2 \times 3 = 12$.

ILLUSTRATED PROBLEM SHEET

Bound a root and test the claim

Squaring a nonnegative number preserves order. The radical sign takes the nonnegative root.

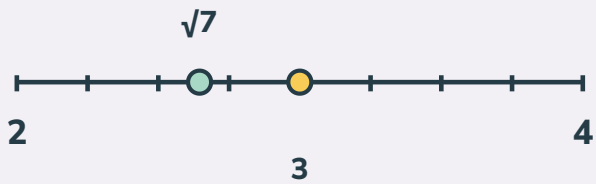
1



Which square has area 49 square units?

Seven rows of seven. Its side length is $\sqrt{49} = 7$.

2



Which integers trap √7?

$2^2 < 7 < 3^2$. Hence $2 < \sqrt{7} < 3$.

3

$$3.3^2 = 10.89$$
$$\downarrow$$
$$11$$
$$\downarrow$$
$$3.4^2 = 11.56$$

Which tenths trap √11?

$3.3 < \sqrt{11} < 3.4$. Compare squares before rounding the root.

4

$$x^2 = 25$$
$$\downarrow$$
$$x = 5 \text{ or } x = -5$$
$$\downarrow$$
$$\sqrt{25} = 5$$

Why are there two equation solutions but one square root symbol value?

Both signs square to 25. The radical denotes the nonnegative one.

$\sqrt{81}$
positive 9.

$x^2 = 81$
two real solutions, 9 and −9.

CHAPTER PROBLEM

Write the √2 proof from the lowest-terms assumption to the contradiction.

Assume $\sqrt{2} = a/b$ in lowest terms, $b > 0$. Derive $a^2 = 2b^2$ and $a = 2k$. Substitute to derive $b^2 = 2k^2$, making b even. Both are even, contradicting coprimality. Conclude irrationality.

QUESTION BANK · FOLD OR COVER THE RIGHT COLUMN

Check your reasoning

Try the problem	Solution and reasoning
01 Which assumption starts a contradiction proof for √2? A) √2 is approximately 1.41 B) √2 = a/b with a, b coprime integers and b > 0 C) a and b are both even D) √2 has no decimal expansion	√2 = a/b with a, b coprime integers and b > 0 Assume rationality and choose lowest terms so a later common factor gives a contradiction.
02 Why does the same contradiction fail for √4?	After $p^2 = 4q^2$, substituting p = 2k gives $k^2 = q^2$. It does not force q to be even.
03 Why must a/b be in lowest terms? A) So a < b B) So a and b are both primes C) So a common prime factor later creates a contradiction D) So b equals 1	So a common prime factor later creates a contradiction Lowest terms means a and b share no prime factor.
04 If $a^2 = 2b^2$ and a = 2k, which equation follows? A) $b^2 = k^2$ B) $b^2 = 2k^2$ C) $b^2 = 4k$ D) $b = 2k^2$	$b^2 = 2k^2$ Substitute to get $4k^2 = 2b^2$, then divide by 2.
05 If 3 divides n^2, why does 3 divide n? A) Because all divisors of a square divide its root B) Because 3 is prime and prime factors of n^2 come from n C) Because $n^2 = 3$ D) Because every integer is divisible by 3	Because 3 is prime and prime factors of n^2 come from n The prime-factor argument works for prime 3. The general claim about all divisors is false, for example 4 divides 2^2 but does not divide 2.
06 Which conclusion is the contradiction in the √3 proof? A) a^2 is positive B) b > 0 C) Both a and b are divisible by 3 despite being coprime D) a and b are integers	Both a and b are divisible by 3 despite being coprime A shared factor 3 conflicts with the lowest-terms assumption.

Number	Power of 2	Power of 3
24	3	1
36	2	2

What is the first shared multiple?

Take the larger exponents. LCM = $2^3 \times 3^2 = 72$.

QUESTION BANK · FOLD OR COVER THE RIGHT COLUMN

Check your reasoning

Try the problem	Solution and reasoning
07 Is $\sqrt{(9 + 16)}$ equal to $\sqrt{9} + \sqrt{16}$?	No. The first is 5. The second is 3 + 4 = 7.
08 Use the same prime-factor reasoning to prove that $\sqrt{5}$ is irrational.	Assume $\sqrt{5} = a/b$ with coprime integers and $b > 0$. Squaring gives $a^2 = 5b^2$. Since 5 is prime, $a = 5k$ for an integer k. Substitute and divide by 5 to obtain $b^2 = 5k^2$, so 5 also divides b. The common factor 5 contradicts lowest terms. Therefore $\sqrt{5}$ is irrational.
09 Why does the same argument not show that $\sqrt{4}$ is irrational?	4 is composite, so 4 dividing a^2 does not imply 4 divides a. For example, $a = 2$ has $a^2 = 4$ but a is not divisible by 4. In fact $\sqrt{4} = 2$, which is rational.
10 Why is 1.41421356 an approximation, rather than a proof about $\sqrt{2}$?	A finite decimal is rational. A calculator only displays finitely many digits of $\sqrt{2}$. Exact divisibility reasoning is needed to exclude all possible fractions.
11 Which argument proves irrationality? A) The digits look random B) A calculator gives many digits C) The rational assumption forces a common factor in a lowest-terms fraction D) The root sign appears	The rational assumption forces a common factor in a lowest-terms fraction The common-factor contradiction excludes every rational representation.

$$108 = 2^2 \times 3^3$$
$$\downarrow$$
$$108 \times 3 = 2^2 \times 3^4$$

Which smallest positive integer makes 108 a square when multiplied by it?

Three supplies the missing partner for the odd exponent. The result is 18^2.

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