MATHS FIELD GUIDE · GRADE 9

Cube and three-term identities

OPENING PROBLEM

Why are the middle coefficients three?

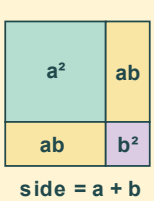
$$(a + b)^3$$
$$\downarrow$$
$$(a^2 + 2ab + b^2)(a + b)$$
$$\downarrow$$
$$a^3 + 3a^2b + 3ab^2 + b^3$$

02 Rules and worked cases
03 Recognise a cube hiding in a product
04 Recognise the conditions
05 From the givens to the answer
06 Match each pattern to its expansion
07 Check your reasoning
08 Check your reasoning

ALGEBRAIC IDENTITIES · FORMULA REFERENCE

Cube and three-term identities

Recognise the form	Use the identity	See it in numbers or symbols
Square of a sum	$(a + b)^2 = a^2 + 2ab + b^2$	$(3x + 2)^2 = 9x^2 + 12x + 4$
Square of a difference	$(a - b)^2 = a^2 - 2ab + b^2$	$(5y - 1)^2 = 25y^2 - 10y + 1$
Difference of squares	$a^2 - b^2 = (a - b)(a + b)$	$49t^2 - 4 = (7t - 2)(7t + 2)$
Two linear factors	$(x + a)(x + b) = x^2 + (a + b)x + ab$	$(x - 3)(x + 8) = x^2 + 5x - 24$
Cube of a sum	$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	$(x + 2)^3 = x^3 + 6x^2 + 12x + 8$
Cube of a difference	$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$	$(2x - 1)^3 = 8x^3 - 12x^2 + 6x - 1$
Sum of cubes	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	$8x^3 + 27 = (2x + 3)(4x^2 - 6x + 9)$
Difference of cubes	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	$27y^3 - 1 = (3y - 1)(9y^2 + 3y + 1)$
Three-term square	$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$	$(x + y + 1)^2 = x^2 + y^2 + 1 + 2xy + 2y + 2x$
Group a common pair	$ax + ay + bx + by = (a + b)(x + y)$	$mn + 2m + 3n + 6 = (m + 3)(n + 2)$



Where does the 2ab come from?

The top-right and bottom-left rectangles each have area ab. Together they supply 2ab. As a check, when a = b = 3, the large square has area 36. The two small squares alone give only 18.

ILLUSTRATED PROBLEM SHEET

Recognise a cube hiding in a product

Multiplying a squared binomial by one more binomial creates four like-term groups.

1

$$(a + b)^3$$
$$\downarrow$$
$$(a^2 + 2ab + b^2)(a + b)$$
$$\downarrow$$
$$a^3 + 3a^2b + 3ab^2 + b^3$$

Why are the middle coefficients three?

The b can come from any one of three brackets, or the a can come from any one bracket.

2

$$(2x - 1)^3$$
$$\downarrow$$
$$8x^3 - 12x^2 + 6x - 1$$

Which signs alternate?

Terms with an odd number of negative ones are negative. At x = 1 the expansion gives 1.

3

$$8x^3 + 27$$
$$\downarrow$$
$$(2x)^3 + 3^3$$
$$\downarrow$$
$$(2x + 3)(4x^2 - 6x + 9)$$

How does the middle sign help?

When the factors multiply, both mixed-degree terms cancel.

4

$$a^3 - b^3$$
$$\downarrow$$
$$(a - b)(a^2 + ab + b^2)$$

How can you check the signs quickly?

Multiply out. The mixed products cancel, leaving the two original cubes.

$(a + b)^3$ at $b = 0$
a^3.

$(a + b)^3$ at $b = a$
$8a^3$.

CHAPTER PROBLEM

Find the coefficient of x^2 in $(x + 2)^3$.

6. The x^2-term is $3 \times x^2 \times 2 = 6x^2$.

RULES AND EXAMPLES

Recognise the conditions

Rules, conditions and worked cases

Recognise the case	Mathematical reasoning
Cube of a sum	$(2x + 1)^3 = 8x^3 + 12x^2 + 6x + 1$. Coefficients follow 1, 3, 3, 1.
Factor a cube sum	$8x^3 + 27 = (2x + 3)(4x^2 - 6x + 9)$. Multiplication cancels the mixed terms.
Square three terms	$(a + b + c)^2$ has the three squares and twice each distinct pair: 2ab, 2bc and 2ca.

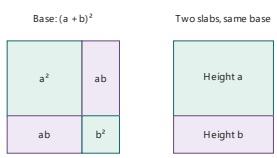
Read the representations

A cube can be split by its height

A cube with side a + b has base area $(a + b)^2$ and height a + b.

Split its height into a and b. Each slab's volume is its height times the same base area.

Combining the slabs gives $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.



Add the slab volumes

Height a	$a^3 + 2a^2b + ab^2$
Height b	$a^2b + 2ab^2 + b^3$

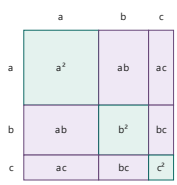
Positive a and b give a geometric volume model. The two slab expansions explain the repeated mixed-volume terms.

Three terms create nine products

To square a + b + c, multiply each term in one bracket by each term in the other.

The three diagonal products are a^2, b^2 and c^2. Each mixed pair appears twice.

Thus $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.



Every cell is one product. The matching off-diagonal cells explain each coefficient 2.

Cubes have their own factor pairs

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

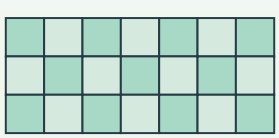
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

Multiplying either pair makes the middle products cancel. These formulas do not come from simply cubing a ± b.

Factors whose mixed products cancel

Original	Factorisation
$a^3 + b^3$	$(a + b)(a^2 - ab + b^2)$
$a^3 - b^3$	$(a - b)(a^2 + ab + b^2)$

The sign outside the quadratic factor is opposite to the sign of its ab term. The b^2 term is positive in both cases.



Split this 3 by 7 array after five columns. What equality appears?

3 × 7 = 3 × 5 + 3 × 2. The two pieces cover exactly the original rectangle.

WORKED EXAMPLES

From the givens to the answer

Is $(a - b)^3$ just $a^3 - b^3$?

A learner drops every mixed product.

1. Substitute $-b$ for b in the addition-cube identity.
2. The correct expansion is $a^3 - 3a^2b + 3ab^2 - b^3$.
3. The alternating signs follow the odd and even powers of $-b$.

Factor $u^3 + 8$ and verify

1. Write $8 = 2^3$, so the sum-of-cubes factors are $(u + 2)(u^2 - 2u + 4)$.
2. Expand: $u^3 - 2u^2 + 4u + 2u^2 - 4u + 8$.
3. The middle products cancel, leaving $u^3 + 8$.

Expand $(x + 2 + y)^2$

1. Use the three square terms: $x^2 + 4 + y^2$.
2. Add twice each distinct pair: $4x + 2xy + 4y$.
3. The expansion is $x^2 + y^2 + 2xy + 4x + 4y + 4$.

Check another case

Example 1

Which factors $8 - w^3$?

$(2 - w)(4 + 2w + w^2) \cdot (2 - w)(4 - 2w + w^2) \cdot (2 - w)^3 \cdot (2 + w)(4 + w^2)$

$(2 - w)(4 + 2w + w^2)$

$8 = 2^3$, so apply the difference-of-cubes factors.

Example 2

Explain why $(a + b + c)^2$ has three square terms and three doubled mixed terms.

Each bracket contributes one of three terms, making nine products.

Three products use the same term twice and are squares.

The remaining six form three equal pairs, producing the doubled mixed terms.

$$4(x + 3)$$
$$\downarrow$$
$$4x + 4 \times 3$$
$$\downarrow$$
$$4x + 12$$

Which term is often missed?

The 3 is also multiplied by 4. At $x = 2$ both sides give 20.

ILLUSTRATED PROBLEM SHEET

Match each pattern to its expansion

Multiplying two sums forms every pairwise product.

1

×	a	b
a	a^2	ab
b	ab	b^2

Where does 2ab in $(a + b)^2$ come from?

The two rectangular off-diagonal products are both ab.

2

$$(3x - 2)^2$$
$$\downarrow$$
$$(3x)^2 - 2(3x)(2) + 2^2$$
$$\downarrow$$
$$9x^2 - 12x + 4$$

Which signs change in a squared difference?

The middle product is negative. The constant is positive because $(-2)^2 = 4$.

3

$$49x^2 - 16$$
$$\downarrow$$
$$(7x)^2 - 4^2$$
$$\downarrow$$
$$(7x - 4)(7x + 4)$$

Why is there no middle term?

The cross-products +28x and −28x cancel.

4

×	x	−3
x	x^2	−3x
5	5x	−15

Collect the products of $(x + 5)(x - 3)$.

The middle terms combine to 2x, giving $x^2 + 2x - 15$.

$(p + q)^2 - (p - q)^2$
4pq.

$(p + q)^2 + (p - q)^2$
$2p^2 + 2q^2$.

CHAPTER PROBLEM

Find the coefficient of x in $(x + 3)^3$.

27. The x-term is $3 \times x \times 3^2 = 27x$.

QUESTION BANK · FOLD OR COVER THE RIGHT COLUMN

Check your reasoning

Try the problem	Solution and reasoning
01 Is $2x^3$ equal to $(2x)^3$?	At x = 1 the first is 2 and the second is 8. Cubing the product cubes both factors.
02 Which factors $t^3 - 27$? A) $(t - 3)(t^2 + 3t + 9)$ B) $(t - 3)(t^2 - 3t + 9)$ C) $(t - 3)^3$ D) $(t + 3)(t^2 + 9)$	$(t - 3)(t^2 + 3t + 9)$ Use $t^3 - 3^3$.
03 Use one identity to calculate 99^2 without long multiplication.	$(100 - 1)^2 = 10{,}000 - 200 + 1 = 9801$.
04 Which is $(x - 1)^3$? A) $x^3 - 1$ B) $x^3 - 3x^2 + 3x - 1$ C) $x^3 + 3x^2 + 3x + 1$ D) $x^3 - 3x^2 - 3x - 1$	$x^3 - 3x^2 + 3x - 1$ Substitute b = 1 in the subtraction-cube identity.
05 Which factors $v^3 + 64$? A) $(v + 4)(v^2 - 4v + 16)$ B) $(v + 4)(v^2 + 4v + 16)$ C) $(v + 4)^3$ D) $(v - 4)(v^2 + 16)$	$(v + 4)(v^2 - 4v + 16)$ $64 = 4^3$ and the sum-of-cubes quadratic has middle term $-4v$.
06 Find the coefficient of xy in $(x + y + 5)^2$.	2 The two products xy and yx combine to 2xy.

5x + 15
↓
5(x + 3)

Can you reverse the split?

Both terms contain 5. Dividing each by 5 leaves x and 3 inside the bracket.

QUESTION BANK · FOLD OR COVER THE RIGHT COLUMN

Check your reasoning

Try the problem	Solution and reasoning
07 Calculate 11^3 using $(10 + 1)^3$.	1331 1000 + 300 + 30 + 1 = 1331.
08 Which expression contains all mixed terms of $(p + q + r)^2$? A) pq + qr + rp B) 2pq + 2qr + 2rp C) 3pqr D) $2p^2 + 2q^2 + 2r^2$	2pq + 2qr + 2rp Every distinct pair occurs in both orders.
09 Explain why the coefficient of a^2b in $(a + b)^3$ is 3.	In the height-a slab, a times 2ab gives $2a^2b$. In the height-b slab, b times a^2 gives another a^2b. Their total is $3a^2b$.
10 Explain how expansion verifies $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.	The a-products are $a^3 + a^2b + ab^2$. The negative b-products are $-a^2b - ab^2 - b^3$. The mixed terms cancel, leaving $a^3 - b^3$.
11 Find the coefficient of x^2 in $(x - 4)^3$.	-12 The term is $-3 \times x^2 \times 4$.

−2(x − 4)
↓
−2x + 8

Why is the constant positive?

It comes from multiplying two negative factors. Test x = 0: both sides give 8.

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