MATHS FIELD GUIDE · GRADE 10

The Pythagorean trigonometric identity

OPENING PROBLEM

Which angles are excluded from this division?

$$\sin^2\theta + \cos^2\theta = 1$$
$$\downarrow$$
$$\text{divide by } \cos^2\theta$$
$$\downarrow$$
$$\tan^2\theta + 1 = \sec^2\theta$$

02 Check the domain before dividing
03 Rules and worked cases
04 From the givens to the answer
05 Choose the sides for this angle
06 Check your reasoning
07 Check your reasoning

ILLUSTRATED PROBLEM SHEET

Check the domain before dividing

Begin with $\sin^2\theta + \cos^2\theta = 1$. Dividing requires a nonzero divisor.

1

$$\sin^2\theta + \cos^2\theta = 1$$
$$\downarrow$$
divide by $\cos^2\theta$
$$\downarrow$$
$$\tan^2\theta + 1 = \sec^2\theta$$

Which angles are excluded from this division?

Angles where cosine equals zero, including 90°.

2

$$\sin\theta = 3/5$$
$$\downarrow$$
$$\cos^2\theta = 16/25$$
$$\downarrow$$
$$\cos\theta = 4/5 \text{ if acute}$$

Where does the acute-angle condition enter?

When choosing the positive square root. The squared relation alone does not determine the sign.

3

$$1 - \cos^2\theta$$
$$\downarrow$$
$$(1 - \cos\theta)(1 + \cos\theta)$$

Simplify $(1 - \cos^2\theta)/(1 - \cos\theta)$.

It equals $1 + \cos\theta$ wherever $\cos\theta \neq 1$. Keep the original restriction.

4

$$(\sin\theta + \cos\theta)^2$$
$$\downarrow$$
$$1 + 2\sin\theta\cos\theta$$

Is the square always one?

At 45° the cross term equals one, giving two. The proposed identity fails.

$\tan^2\theta + 1 = \sec^2\theta$ requires $\cos\theta \neq 0$.

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$ requires $\sin\theta \neq 0$.

CHAPTER PROBLEM

For acute θ, $\sin\theta = 5/13$. Find $\cos\theta$.

12/13. $\cos^2\theta = 1 - 25/169 = 144/169$ and $\cos\theta > 0$.

MATHS REFERENCE · GRADE 10

The Pythagorean trigonometric identity

Rules, conditions and worked cases

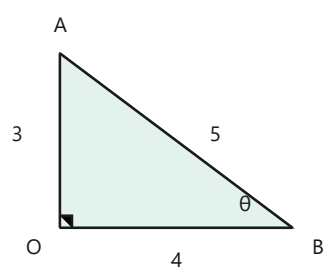
Recognise the case	Mathematical reasoning
From a triangle	Divide $a^2 + b^2 = c^2$ by c^2. The squared leg ratios give $\sin^2\theta + \cos^2\theta = 1$.
Recover a ratio	For acute θ with $\sin\theta = 5/13$, $\cos^2\theta = 144/169$. Take the positive root $12/13$.
Derived identities	Dividing by $\cos^2\theta$ gives $1 + \tan^2\theta = \sec^2\theta$ where $\cos\theta \neq 0$. Dividing by $\sin^2\theta$ gives $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ where $\sin\theta \neq 0$.

Read the representations

Divide Pythagoras by the same nonzero square

In a right triangle relative to acute θ, let opposite=o, adjacent=a and hypotenuse=h. Since $o^2+a^2=h^2$, dividing by h^2 gives $(o/h)^2+(a/h)^2=1$.

Therefore $\sin^2\theta+\cos^2\theta=1$. For real angles where defined, it expresses a relationship between ratios, not $\sin(\theta^2)$.



$o^2 + a^2 = h^2$; divide by h^2.

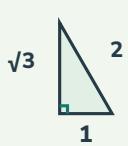
Derived identities have denominator conditions

Divide by $\cos^2\theta$ only when $\cos\theta\neq0$ to obtain $\tan^2\theta+1=\sec^2\theta$.

Divide by $\sin^2\theta$ only when $\sin\theta\neq0$ to obtain $1+\cot^2\theta=\operatorname{cosec}^2\theta$. In the acute-angle setting both conditions hold.

Same relationship, different denominator

Divide by	Result	Restriction
1	$\sin^2\theta+\cos^2\theta=1$	Both ratios defined
$\cos^2\theta$	$1+\tan^2\theta=\sec^2\theta$	$\cos\theta\neq0$
$\sin^2\theta$	$1+\cot^2\theta=\operatorname{cosec}^2\theta$	$\sin\theta\neq0$



Which acute angle faces the short leg?

30°. Thus $\sin 30° = 1/2$ and $\cos 30° = \sqrt{3}/2$.

WORKED EXAMPLES

From the givens to the answer

Recover a missing ratio with the right sign

1. For acute θ, $\sin\theta=3/5$, so $\cos^2\theta=1-9/25=16/25$.
2. Since θ is acute, $\cos\theta>0$, hence $\cos\theta=4/5$.
3. Then $\tan\theta=(3/5)/(4/5)=3/4$. The domain supplies the positive sign.

Recover a ratio and choose its sign

Angle θ is acute and $\cos\theta = 12/13$. Find $\sin\theta$ and $\tan\theta$.

1. The identity gives $\sin^2\theta = 1 - 144/169 = 25/169$. An acute angle has positive sine, so $\sin\theta = 5/13$. Dividing by $12/13$ gives $\tan\theta = 5/12$.

Check another case

Example 1

If $\text{cosec}^2\theta=11$, find $\cot^2\theta$.

10

$\cot^2\theta=\text{cosec}^2\theta-1$.

Example 2

If $\tan^2\theta=8$, find $\sec^2\theta$.

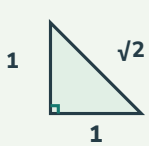
9

$\sec^2\theta=1+\tan^2\theta$.

Spot two ratios that cannot belong together

A pupil reports $\sin\theta = 3/5$ and $\cos\theta = 3/4$ for the same angle. Their squared sum is $9/25 + 9/16 = 369/400$.

It must equal one. For an acute angle with sine 3/5, the matching cosine is 4/5.



Why is tan 45° exactly one?

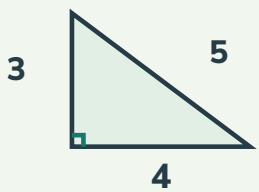
The legs match. Sine and cosine each equal $1/\sqrt{2}$.

ILLUSTRATED PROBLEM SHEET

Choose the sides for this angle

In a right triangle the hypotenuse stays fixed. Opposite and adjacent swap when the other acute angle is chosen.

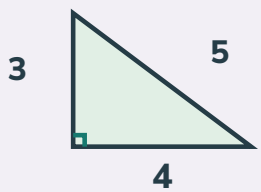
1



For the acute angle opposite 3, find sine and cosine.

sin θ = 3/5 and cos θ = 4/5. The hypotenuse is the denominator in each.

2



Use the angle opposite 4 instead. What happens to tangent?

It becomes 4/3. For the other angle it was 3/4.

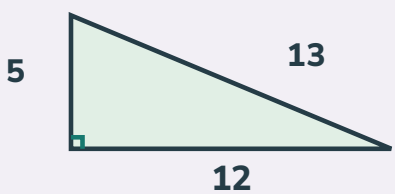
3

$$\sin \theta = 5/13$$
$$\downarrow$$
$$\text{cosec } \theta = 13/5$$

Why does the fraction invert?

Cosecant is hypotenuse over opposite. Its product with sine is one wherever both are defined.

4



Use the triangle to check $\sin^2 \theta + \cos^2 \theta$.

The numerator is 25 + 144 = 169 over 169. The sum is one.

Opposite 5, adjacent 12
tan θ = 5/12.

Same triangle, other acute angle
tan = 12/5.

CHAPTER PROBLEM

If $\tan^2\theta=4$, find $\sec^2\theta$.

5. $\sec^2\theta=1+\tan^2\theta$.

QUESTION BANK · FOLD OR COVER THE RIGHT COLUMN

Check your reasoning

Try the problem	Solution and reasoning
01 If sec θ + tan θ = 4, find sec θ − tan θ.	Their product is sec² θ − tan² θ = 1. The second factor is 1/4.
02 Could 7/5 be the sine of an acute angle?	No. Its opposite side would exceed the hypotenuse. A reciprocal ratio such as cosecant can exceed one.
03 If $\text{cosec}^2\theta=9$, find $\cot^2\theta$.	8 $\cot^2\theta=\text{cosec}^2\theta-1$.
04 When can $\sin^2\theta+\cos^2\theta=1$ be divided by $\sin^2\theta$? A) Always, including $\sin\theta=0$ B) Only when $\sin\theta\neq0$ C) Only when $\cos\theta=0$ D) Never	Only when $\sin\theta\neq0$ Division by zero is not permitted.
05 For acute θ, $\cos\theta=8/17$. Find $\sin\theta$.	15/17 $\sin^2\theta=1-64/289=225/289$; choose positive root.
06 Evaluate $\sin^2 37°+\cos^2 37°$.	1 The identity applies without finding either individual ratio.

θ	0°	90°
cos θ	1	0
sin θ	0	1

At which endpoint does tan θ fail?

At 90°. Cosine is zero, so the quotient sine/cosine is undefined.

QUESTION BANK · FOLD OR COVER THE RIGHT COLUMN

Check your reasoning

Try the problem	Solution and reasoning
07 Derive $1+\tan^2\theta=\sec^2\theta$ and state its restriction.	Start with $\sin^2\theta+\cos^2\theta=1$. Divide through by $\cos^2\theta$ with $\cos\theta\neq0$, then use the quotient and reciprocal definitions.
08 Why is $\sin^2\theta$ shorthand for $(\sin\theta)^2$ rather than $\sin(\theta^2)$?	The exponent applies to the ratio value. The angle remains θ; moving the exponent inside changes the function input.
09 If $\tan^2\theta=5$, find $\sec^2\theta$.	6 $\sec^2\theta=1+\tan^2\theta$.
10 If $\tan^2\theta=6$, find $\sec^2\theta$.	7 $\sec^2\theta=1+\tan^2\theta$.
11 If $\tan^2\theta=7$, find $\sec^2\theta$.	8 $\sec^2\theta=1+\tan^2\theta$.

$\theta = 60°$
↓
$\sin^2\theta = 3/4$
↓
$\cos^2\theta = 1/4$

Check the identity at an exact angle.

Three quarters plus one quarter equals one. The individual ratios are square roots of these values.

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