MATHS FIELD GUIDE · GRADE 10

Simple identity proofs and error analysis

OPENING PROBLEM

Which angles are excluded from this division?

$$\sin^2\theta + \cos^2\theta = 1$$
$$\downarrow$$
$$\text{divide by } \cos^2\theta$$
$$\downarrow$$
$$\tan^2\theta + 1 = \sec^2\theta$$

02 Check the domain before dividing
03 Rules and worked cases
04 From the givens to the answer
05 Rebuild the exact trigonometric values
06 Check your reasoning
07 Check your reasoning

ILLUSTRATED PROBLEM SHEET

Check the domain before dividing

Begin with $\sin^2\theta + \cos^2\theta = 1$. Dividing requires a nonzero divisor.

1

$$\sin^2\theta + \cos^2\theta = 1$$
$$\downarrow$$
divide by $\cos^2\theta$
$$\downarrow$$
$$\tan^2\theta + 1 = \sec^2\theta$$

Which angles are excluded from this division?

Angles where cosine equals zero, including 90°.

2

$$\sin\theta = 3/5$$
$$\downarrow$$
$$\cos^2\theta = 16/25$$
$$\downarrow$$
$$\cos\theta = 4/5 \text{ if acute}$$

Where does the acute-angle condition enter?

When choosing the positive square root. The squared relation alone does not determine the sign.

3

$$1 - \cos^2\theta$$
$$\downarrow$$
$$(1 - \cos\theta)(1 + \cos\theta)$$

Simplify $(1 - \cos^2\theta)/(1 - \cos\theta)$.

It equals $1 + \cos\theta$ wherever $\cos\theta \neq 1$. Keep the original restriction.

4

$$(\sin\theta + \cos\theta)^2$$
$$\downarrow$$
$$1 + 2\sin\theta\cos\theta$$

Is the square always one?

At 45° the cross term equals one, giving two. The proposed identity fails.

$\tan^2\theta + 1 = \sec^2\theta$ requires $\cos\theta \neq 0$.

$1 + \cot^2\theta = \operatorname{cosec}^2\theta$ requires $\sin\theta \neq 0$.

CHAPTER PROBLEM

If $\sec\theta+\tan\theta=4$, find $\sec\theta-\tan\theta$.

0.25. The product of the two expressions is 1.

MATHS REFERENCE · GRADE 10

Simple identity proofs and error analysis

Rules, conditions and worked cases

Recognise the case	Mathematical reasoning
Use a conjugate	$(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = \sec^2\theta - \tan^2\theta = 1$ where $\sec\theta$ is defined.
Simplify by factoring	$(1 - \cos^2\theta)/\sin\theta = \sin^2\theta/\sin\theta = \sin\theta$ for $\sin\theta \neq 0$.
Disprove a false identity	At 45°, $\sin\theta + \cos\theta = \sqrt{2}$, so it cannot always equal one. A single counterexample suffices.

Read the representations

An identity must work throughout its common domain

Transform one side using definitions, factorisation and identities until it matches the other.

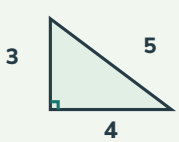
Checking one numerical angle cannot prove a statement for all permitted angles. Record denominator restrictions before cancelling.

1. **Start from one side** $(1-\cos^2\theta)/\sin\theta$
2. **Use Pythagoras** $\sin^2\theta/\sin\theta$
3. **Cancel with a condition** $\sin\theta$, provided $\sin\theta \neq 0$

Conjugates make a hidden factor visible

For $\sin\theta \neq 0$ and $1+\cos\theta \neq 0$, $\sin\theta/(1+\cos\theta) = (1-\cos\theta)/\sin\theta$.

Multiply numerator and denominator by $1-\cos\theta$ or cross-multiply over the common domain. An equality may have different-looking denominators; both sides must be defined.



For the acute angle opposite 3, find sine and cosine.

$\sin\theta = 3/5$ and $\cos\theta = 4/5$. The hypotenuse is the denominator in each.

WORKED EXAMPLES

From the givens to the answer

A conjugate product equals one

1. Start with $(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)$.
2. Multiply the factors: $\sec^2\theta - \tan^2\theta$.
3. Use $\sec^2\theta = 1 + \tan^2\theta$, giving 1.
4. The original factors require $\cos\theta \neq 0$.

Cancel a common factor

1. To prove $(1 - \cos^2\theta)/(1 + \cos\theta) = 1 - \cos\theta$, factor the numerator as $(1 - \cos\theta)(1 + \cos\theta)$.
2. Cancel the nonzero factor $1 + \cos\theta$.
3. The equality holds when $\cos\theta \neq -1$; the unreduced fraction is undefined at $\cos\theta = -1$.

Repair a tempting cancellation

Can $(1 - \cos^2\theta)/(1 - \cos\theta)$ always be replaced by $1 + \cos\theta$?

1. Factor the numerator into $(1 - \cos\theta)(1 + \cos\theta)$. Cancellation is valid only when $\cos\theta \neq 1$. At $\theta = 0°$ the original quotient is 0/0 and has no value.

Check another case

Example 1

If $\cot^2\theta = 3$, evaluate $\operatorname{cosec}^2\theta - \cot^2\theta$.

1

Use the Pythagorean reciprocal identity.

Example 2

If $\sec\theta + \tan\theta = 8$, find $\sec\theta - \tan\theta$.

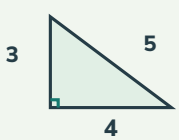
0.125

The product of the two expressions is 1.

Test an alleged shortcut

Does $(\sin\theta + \cos\theta)^2$ always equal 1? Expand it: the result is $1 + 2\sin\theta\cos\theta$.

At 45° the cross term equals one and the square equals two. Dropping the cross term produces a false identity.



Use the angle opposite 4 instead. What happens to tangent?

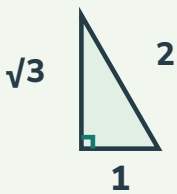
It becomes 4/3. For the other angle it was 3/4.

ILLUSTRATED PROBLEM SHEET

Rebuild the exact trigonometric values

Use half an equilateral triangle or half a square. At 0° and 90° use unit-circle coordinates.

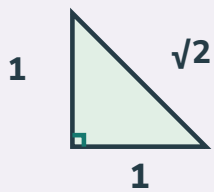
1



Which acute angle faces the short leg?

30°. Thus sin 30° = 1/2 and cos 30° = √3/2.

2



Why is tan 45° exactly one?

The legs match. Sine and cosine each equal 1/√2.

3

θ	0°	90°
cos θ	1	0
sin θ	0	1

At which endpoint does tan θ fail?

At 90°. Cosine is zero, so the quotient sine/cosine is undefined.

4

$$\theta = 60°$$
$$\downarrow$$
$$\sin^2\theta = 3/4$$
$$\downarrow$$
$$\cos^2\theta = 1/4$$

Check the identity at an exact angle.

Three quarters plus one quarter equals one. The individual ratios are square roots of these values.

sin 30° equals cos 60°
both 1/2.

sin 60° equals cos 30°
both √3/2.

CHAPTER PROBLEM

For acute θ, sinθ=3/5. Evaluate (1−cos²θ)/sinθ.

3/5. The expression reduces to sinθ.

QUESTION BANK · FOLD OR COVER THE RIGHT COLUMN

Check your reasoning

Try the problem	Solution and reasoning
01 If $\sec\theta + \tan\theta = 4$, find $\sec\theta - \tan\theta$.	Their product is $\sec^2\theta - \tan^2\theta = 1$. The second factor is 1/4.
02 Is cosec 0° equal to zero?	No. It is $1/\sin 0°$, so the denominator is zero and it is undefined.
03 Which proves an identity? A) Checking $\theta=45°$ only B) A chain of valid transformations on the common domain C) Rounding three calculator examples D) Drawing one triangle without reasons	A chain of valid transformations on the common domain All permitted inputs must be covered.
04 If $\cot^2\theta=8$, evaluate $\operatorname{cosec}^2\theta-\cot^2\theta$.	1 Use the Pythagorean reciprocal identity.
05 For acute θ, $\cos\theta=4/5$ and $\sin\theta=3/5$. Evaluate $\sin\theta/(1+\cos\theta)$.	1/3 $(3/5)/(9/5)=1/3$.
06 Evaluate $(1-\sin^2 30°)/\cos^2 30°$.	1 The numerator equals $\cos^2 30°$, which is nonzero.

sin θ = 5/13
↓
cosec θ = 13/5

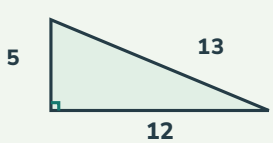
Why does the fraction invert?

Cosecant is hypotenuse over opposite. Its product with sine is one wherever both are defined.

QUESTION BANK · FOLD OR COVER THE RIGHT COLUMN

Check your reasoning

Try the problem	Solution and reasoning
07 Prove $(\operatorname{cosec}\theta-\cot\theta)(\operatorname{cosec}\theta+\cot\theta)=1$.	Expand as $\operatorname{cosec}^2\theta-\cot^2\theta$. Use $1+\cot^2\theta=\operatorname{cosec}^2\theta$; state $\sin\theta\neq0$.
08 A learner cancels $\sin\theta$ in $\sin^2\theta/\sin\theta$ without stating a condition. Explain the repair.	Require $\sin\theta\neq0$ for the starting fraction to be defined. The simplified form $\sin\theta$ alone is defined more widely, but that does not restore excluded inputs to the original.
09 If $\sec\theta+\tan\theta=5$, find $\sec\theta-\tan\theta$.	0.2 The product of the two expressions is 1.
10 If $\sec\theta+\tan\theta=6$, find $\sec\theta-\tan\theta$.	1/6 Their product is $\sec^2\theta-\tan^2\theta=1$. The missing factor is exactly 1/6.
11 If $\sec\theta+\tan\theta=7$, find $\sec\theta-\tan\theta$.	1/7 The product $(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)$ equals $\sec^2\theta-\tan^2\theta=1$. The missing factor is exactly 1/7.



Use the triangle to check $\sin^2\theta+\cos^2\theta$.

The numerator is 25 + 144 = 169 over 169. The sum is one.

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