MATHS FIELD GUIDE · GRADE 10

Proving square roots irrational

OPENING PROBLEM

Why must p be even?

$$\sqrt{2} = p/q$$
$$\downarrow$$
$$p^2 = 2q^2$$
$$\downarrow$$
$$p \text{ is even}$$

02 Try the fraction and follow the contradiction
03 Rules and worked cases
04 From the givens to the answer
05 Use the prime powers as building blocks
06 Check your reasoning

ILLUSTRATED PROBLEM SHEET

Try the fraction and follow the contradiction

A rational number can be written as p/q in lowest terms, with q nonzero.

1

$$\sqrt{2} = \frac{p}{q}$$
$$\downarrow$$
$$p^2 = 2q^2$$
$$\downarrow$$
$$p \text{ is even}$$

Why must p be even?

An odd integer has an odd square. An even square can only come from an even integer.

2

$$p = 2k$$
$$\downarrow$$
$$4k^2 = 2q^2$$
$$\downarrow$$
$$q^2 = 2k^2$$

What does this imply about q?

It is even too. Both p and q divisible by 2 contradict lowest terms.

3

$$r = 3 + \sqrt{2}$$
$$\downarrow$$
$$\sqrt{2} = r - 3$$

Could r be rational?

If so, subtracting the rational number 3 would make √2 rational. That contradicts the proof.

4

$$(\sqrt{2})(\sqrt{2}) = 2$$
$$\downarrow$$
$$\sqrt{2} - \sqrt{2} = 0$$

Can two irrational inputs produce a rational result?

Yes. Both calculations do. The type of each input alone does not decide the output.

$\sqrt{18} = 3\sqrt{2}$
irrational.

$\sqrt{18} \div \sqrt{2} = 3$
rational.

CHAPTER PROBLEM

Write a full proof that √3 is irrational.

Assume $\sqrt{3} = a/b$ in lowest terms; $a^2 = 3b^2$. 3 divides a, so $a = 3k$; substitution gives $b^2 = 3k^2$. 3 divides b, contradicting coprimality. Conclude irrationality.

MATHS REFERENCE · GRADE 10

Proving square roots irrational

Rules, conditions and worked cases

Recognise the case	Mathematical reasoning
Set up the contradiction	Write $\sqrt{3} = a/b$ with a and b coprime, $b > 0$. Then $a^2 = 3b^2$.
Force a shared prime	3 divides a^2, hence a. Put $a = 3k$ to obtain $b^2 = 3k^2$. Now both contain 3.
Test the boundary	$\sqrt{9} = 3$. The step "9 divides a^2, so 9 divides a" fails for $a = 3$ because 9 is composite.

Read the representations

A rational number has an exact integer ratio

Rational means a/b with integers a and b, $b \neq 0$. Reduce the ratio so HCF(a,b) = 1.

An irrational real number has no such exact ratio. A rounded decimal cannot decide rationality.

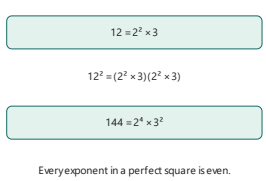
Exact values, different types

Value	Reason
0.125 = 1/8	Rational
√49 = 7/1	Rational
√2	Irrational; requires a proof

Squaring pairs every prime factor

In n^2, each exponent in the prime factorisation is even.

If a prime p divides n^2, then p divides n. For example, an odd square cannot have a factor 2.

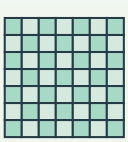


Contradiction is a logical test, not a numerical estimate

Assume a reduced ratio for the root. Squaring forces the same prime to divide both integers.

That contradicts the starting condition that the ratio was in lowest terms.

1. **Assume the opposite** $\sqrt{p} = a/b$ in lowest terms.
2. **Use prime divisibility** $a^2 = pb^2$ forces p to divide a.
3. **Substitute a = pk** $b^2 = pk^2$ forces p to divide b.
4. **Contradiction** a and b cannot both share p in lowest terms.



Which square has area 49 square units?

Seven rows of seven. Its side length is $\sqrt{49} = 7$.

WORKED EXAMPLES

From the givens to the answer

Prove √2 is irrational

1. Suppose $\sqrt{2} = a/b$ with positive coprime integers a,b. Then $a^2 = 2b^2$.
2. 2 divides a^2, so 2 divides a. Write $a = 2k$.
3. Substitute: $4k^2 = 2b^2$, hence $b^2 = 2k^2$. Thus 2 divides b.
4. Both a and b are even, contradicting coprimality. Therefore √2 is irrational.

Check another case

Example 1

A learner writes $a^2 = 2b^2$, hence both a and b are even, and stops. Supply the missing argument.

Since 2 divides a^2, write $a = 2k$.

Substitution gives $b^2 = 2k^2$, which forces b even.

State the conflict with the coprime starting assumption.

Example 2

Write a full proof that √5 is irrational.

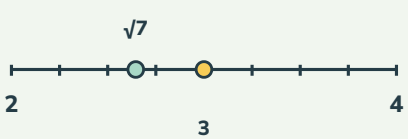
Assume $\sqrt{5} = a/b$ in lowest terms; $a^2 = 5b^2$.

5 divides a, so $a = 5k$; then $b^2 = 5k^2$.

5 divides b contradicts coprimality, so √5 is irrational.

Work backwards and test the limits

Problem	Derivation
Why is "its decimal has many digits" insufficient?	The rational number 1/7 has infinitely many repeating digits. A finite display cannot show that a decimal never becomes periodic.
Prove √12 is irrational using √3.	$\sqrt{12} = 2\sqrt{3}$. If √12 were rational, dividing by 2 would make √3 rational, contradicting its prime-divisibility proof.
Could √8 equal 2.828 exactly?	No. 2.828 is a rational decimal and its square is 7.997584. It estimates the positive root of 8 without equalling it.



Which integers trap √7?

$2^2 < 7 < 3^2$. Hence $2 < \sqrt{7} < 3$.

ILLUSTRATED PROBLEM SHEET

Use the prime powers as building blocks

Prime exponents record how many copies of each prime a number contains.

1

$$72 = 8 \times 9$$
$$\downarrow$$
$$2^3 \times 3^2$$

Why do two different factor trees end alike?

Their terminal prime factors, counted with multiplicity, are unique apart from order.

2

Number	Power of 2	Power of 3
24	3	1
36	2	2

Read the HCF from the exponents.

Take the smaller exponent in each column. HCF = $2^2 \times 3 = 12$.

3

Number	Power of 2	Power of 3
24	3	1
36	2	2

What is the first shared multiple?

Take the larger exponents. LCM = $2^3 \times 3^2 = 72$.

4

$$108 = 2^2 \times 3^3$$
$$\downarrow$$
$$108 \times 3 = 2^2 \times 3^4$$

Which smallest positive integer makes 108 a square when multiplied by it?

Three supplies the missing partner for the odd exponent. The result is 18^2.

$180 = 2^2 \times 3^2 \times 5$
multiply by 5 for a square.

$180 \div 5 = 36$
a square already.

CHAPTER PROBLEM

If $a = 5k$ and $a^2 = 5b^2$, then $b^2 = ck^2$. Find c.

5. $25k^2 = 5b^2$; divide by 5.

QUESTION BANK · FOLD OR COVER THE RIGHT COLUMN

Check your reasoning

Try the problem	Solution and reasoning
01 Why must a/b be chosen in lowest terms in an irrationality proof? A) To make a and b equal. B) To ensure a and b share no prime factor. C) To force b = 1. D) To estimate the root.	To ensure a and b share no prime factor. The contradiction is that a common prime factor is forced.
02 If 3 divides an integer n^2, what follows? A) 3 divides n. B) n = 3. C) 9 divides n. D) n is odd.	3 divides n. Prime divisibility of a square implies divisibility of its base.
03 Why does the same contradiction fail for √4?	After $p^2 = 4q^2$, substituting p = 2k gives $k^2 = q^2$. It does not force q to be even.
04 Why is 1 excluded from the primes?	It has only one positive divisor. Allowing extra factors of 1 would also make prime-factor lists non-unique.
05 Why does the same contradiction argument fail for √4?	From $a^2 = 4b^2$ we obtain a = 2b for positive integers. There is no forced common factor of a and b: a = 2,b = 1 works. √4 = 2.
06 Which is sufficient evidence that √7 is irrational? A) A calculator shows many decimals. B) A rounded value is not a fraction. C) A valid contradiction using prime divisibility. D) It lies between 2 and 3.	A valid contradiction using prime divisibility. Finite decimal evidence and location between integers are insufficient.

$3.3^2 = 10.89$
↓
11
↓
$3.4^2 = 11.56$

Which tenths trap √11?

3.3 < √11 < 3.4. Compare squares before rounding the root.

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