Find an invariant and prove termination

Write your reasoning in the space below. Turn the page when you are ready to check.

Subtract the smaller number

(33, 21) -> (12, 21) -> (12, 9) -> your next move

1. A move on two numbers

Start with 33 and 21. Replace the larger by its difference with the smaller, stopping when they agree. What common number do you reach?

2. What survives a move?

For any two unequal positive integers a > b, prove that (a,b) and (a - b,b) have exactly the same common divisors.

3. Why must it finish?

Prove that this rule reaches two equal numbers from every pair of positive integers. Identify that number in terms of the starting pair.

Worked answers

Find an invariant and prove termination

1. A move on two numbers

Write both entries after each move.

The pairs are (33,21), (12,21), (12,9), (3,9), (3,6), (3,3). The common number is 3. The order of the entries does not affect the move.

2. What survives a move?

Show each direction separately.

If d divides a and b, it divides a - b. Conversely, if d divides a - b and b, it divides their sum a. Because the common divisors coincide, their greatest member is unchanged.

3. Why must it finish?

Watch the sum at each move. Can it fall forever?

For a > b, the sum falls from a + b to a, decreasing by b. It stays a positive integer, so it cannot decrease indefinitely. Each unequal pair allows a move; the process must therefore stop at (g,g). Its greatest common divisor is g, the same as at the start.